

BROUWER'S FIXED POINT THEOREM WITH SEQUENTIALLY AT MOST ONE FIXED POINT

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ABSTRACT. We present a constructive proof of Brouwer's fixed point theorem with sequentially at most one fixed point, and apply it to the mini-max theorem of zero-sum games.

1. INTRODUCTION

It is well known that Brouwer's fixed point theorem can not be constructively proved¹. Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors have presented an approximate version of Brouwer's theorem using Sperner's lemma. See [8] and [9]. Thus, Brouwer's fixed point theorem is constructively, in the sense of constructive mathematics á la Bishop, proved in its approximate version.

Also in [8] Dalen states a conjecture that a uniformly continuous function f from a simplex to itself, with property that each open set contains a point x such that $x \neq f(x)$, which means $|x - f(x)| > 0$, and also at every point x on the boundaries of the simplex $x \neq f(x)$, has an exact fixed point. In this note we present a partial answer to Dalen's conjecture.

Recently [2] showed that the following theorem is equivalent to Brouwer's fan theorem.

Each uniformly continuous function φ from a compact metric space X into itself with at most one fixed point and approximate fixed points has a fixed point.

By reference to the notion of *sequentially at most one maximum* in [1] we require a stronger condition that a function φ has *sequentially at most one fixed point*, and will show the following result.

Each uniformly continuous function φ from a compact metric space X into itself with *sequentially at most one fixed point* and approximate fixed points has a fixed point,

without the fan theorem. In [7] Orevkov constructed a computably coded continuous function f from the unit square to itself, which is defined at each computable point of the square, such that f has no computable fixed point. His map consists of a retract of the computable elements of the square to its boundary followed by a

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¹[6] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics á la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive. See [3] or [8].

rotation of the boundary of the square. As pointed out by Hirst in [5], since there is no retract of the square to its boundary, his map does not have a total extension.

In the next section we present our theorem and its proof. In Section 3, as an application of the theorem we consider the mini-max theorem of two-person zero-sum games.

2. THEOREM AND PROOF

Let \mathbf{p} be a point in a compact metric space X , and consider a uniformly continuous function φ from X into itself. According to [8] and [9] φ has an approximate fixed point. It means

For each $\varepsilon > 0$ there exists $\mathbf{p} \in X$ such that $|\mathbf{p} - \varphi(\mathbf{p})| < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary,

$$\inf_{\mathbf{p} \in X} |\mathbf{p} - \varphi(\mathbf{p})| = 0.$$

The notion that φ has at most one fixed point is defined as follows;

Definition 1 (At most one fixed point). *For all $\mathbf{p}, \mathbf{q} \in X$, if $\mathbf{p} \neq \mathbf{q}$, then $\varphi(\mathbf{p}) \neq \mathbf{p}$ or $\varphi(\mathbf{q}) \neq \mathbf{q}$.*

Next by reference to the notion of *sequentially at most one maximum* in [1], we define the notion that φ has *sequentially at most one fixed point* as follows;

Definition 2 (Sequentially at most one fixed point). *All sequences $(\mathbf{p}_n)_{n \geq 1}$, $(\mathbf{q}_n)_{n \geq 1}$ in X such that $|\varphi(\mathbf{p}_n) - \mathbf{p}_n| \rightarrow 0$ and $|\varphi(\mathbf{q}_n) - \mathbf{q}_n| \rightarrow 0$ are eventually close in the sense that $|\mathbf{p}_n - \mathbf{q}_n| \rightarrow 0$.*

Now we show the following lemma.

Lemma 1. *Let φ be a uniformly continuous function from a compact metric space X into itself. Assume $\inf_{\mathbf{p} \in X} |\mathbf{p} - \varphi(\mathbf{p})| = 0$. If the following property holds,*

For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $\mathbf{p}, \mathbf{q} \in X$, $|\varphi(\mathbf{p}) - \mathbf{p}| < \varepsilon$ and $|\varphi(\mathbf{q}) - \mathbf{q}| < \varepsilon$, then $|\mathbf{p} - \mathbf{q}| \leq \delta$,

then, there exists a point $\mathbf{r} \in X$ such that $\varphi(\mathbf{r}) = \mathbf{r}$, that is, we have a fixed point of φ .

Proof. Choose a sequence $(\mathbf{p}_n)_{n \geq 1}$ in X such that $|\varphi(\mathbf{p}_n) - \mathbf{p}_n| \rightarrow 0$. Compute N such that $|\varphi(\mathbf{p}_n) - \mathbf{p}_n| < \varepsilon$ for all $n \geq N$. Then, for $m, n \geq N$ we have $|\mathbf{p}_m - \mathbf{p}_n| \leq \delta$. Since $\delta > 0$ is arbitrary, $(\mathbf{p}_n)_{n \geq 1}$ is a Cauchy sequence in X , and converges to a limit $\mathbf{r} \in X$. The continuity of φ yields $|\varphi(\mathbf{r}) - \mathbf{r}| = 0$, that is, $\varphi(\mathbf{r}) = \mathbf{r}$. \square

Next we show the following theorem.

Theorem 1. *Each uniformly continuous function φ from a compact metric space X into itself with sequentially at most one fixed point and approximate fixed points has a fixed point*

Proof. Choose a sequence $(\mathbf{r}_n)_{n \geq 1}$ in X such that $|\varphi(\mathbf{r}_n) - \mathbf{r}_n| \rightarrow 0$. In view of Lemma 1 it is enough to prove that the following condition holds.

For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $\mathbf{p}, \mathbf{q} \in X$, $|\varphi(\mathbf{p}) - \mathbf{p}| < \varepsilon$ and $|\varphi(\mathbf{q}) - \mathbf{q}| < \varepsilon$, then $|\mathbf{p} - \mathbf{q}| \leq \delta$.

Assume that the set

$$K = \{(\mathbf{p}, \mathbf{q}) \in X \times X : |\mathbf{p} - \mathbf{q}| \geq \delta\}$$

is nonempty and compact². Since the mapping $(\mathbf{p}, \mathbf{q}) \rightarrow \max(|\varphi(\mathbf{p}) - \mathbf{p}|, |\varphi(\mathbf{q}) - \mathbf{q}|)$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_n)_{n \geq 1}$ such that

$$\lambda_n = 0 \Rightarrow \inf_{(\mathbf{p}, \mathbf{q}) \in K} \max(|\varphi(\mathbf{p}) - \mathbf{p}|, |\varphi(\mathbf{q}) - \mathbf{q}|) < 2^{-n},$$

$$\lambda_n = 1 \Rightarrow \inf_{(\mathbf{p}, \mathbf{q}) \in K} \max(|\varphi(\mathbf{p}) - \mathbf{p}|, |\varphi(\mathbf{q}) - \mathbf{q}|) > 2^{-n-1}.$$

It suffices to find n such that $\lambda_n = 1$. In that case, if $|\varphi(\mathbf{p}) - \mathbf{p}| < 2^{-n-1}$, $|\varphi(\mathbf{q}) - \mathbf{q}| < 2^{-n-1}$, we have $(\mathbf{p}, \mathbf{q}) \notin K$ and $|\mathbf{p} - \mathbf{q}| \leq \delta$. Assume $\lambda_1 = 0$. If $\lambda_n = 0$, choose $(\mathbf{p}_n, \mathbf{q}_n) \in K$ such that $\max(|\varphi(\mathbf{p}_n) - \mathbf{p}_n|, |\varphi(\mathbf{q}_n) - \mathbf{q}_n|) < 2^{-n}$, and if $\lambda_n = 1$, set $\mathbf{p}_n = \mathbf{q}_n = \mathbf{r}_n$. Then, $|\varphi(\mathbf{p}_n) - \mathbf{p}_n| \rightarrow 0$ and $|\varphi(\mathbf{q}_n) - \mathbf{q}_n| \rightarrow 0$, so $|\mathbf{p}_n - \mathbf{q}_n| \rightarrow 0$. Computing N such that $|\mathbf{p}_N - \mathbf{q}_N| < \delta$, we must have $\lambda_N = 1$. We have completed the proof. \square

3. APPLICATION: MINIMAX THEOREM OF ZERO-SUM GAMES

Consider a two person zero-sum game. There are two players A and B . Player A has m alternative pure strategies, and the set of his pure strategies is denoted by $S_A = \{a_1, a_2, \dots, a_m\}$. Player B has n alternative pure strategies, and the set of his pure strategies is denoted by $S_B = \{b_1, b_2, \dots, b_n\}$. m and n are finite natural numbers. The payoff of player A when a combination of players' strategies is (a_i, b_j) is denoted by $M(a_i, b_j)$. Since we consider a zero-sum game, the payoff of player B is equal to $-M(a_i, b_j)$. Let p_i be a probability that A chooses his strategy a_i , and q_j be a probability that B chooses his strategy b_j . A mixed strategy of A is represented by a probability distribution over S_A , and is denoted by $\mathbf{p} = (p_1, p_2, \dots, p_m)$ with $\sum_{i=1}^m p_i = 1$. Similarly, a mixed strategy of B is denoted by $\mathbf{q} = (q_1, q_2, \dots, q_n)$ with $\sum_{j=1}^n q_j = 1$. A combination of mixed strategies (\mathbf{p}, \mathbf{q}) is called a *profile*. The expected payoff of player A at a profile (\mathbf{p}, \mathbf{q}) is written as follows,

$$M(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^m \sum_{j=1}^n p_i M(a_i, b_j) q_j.$$

We assume that $M(a_i, b_j)$ is finite. Then, since $M(\mathbf{p}, \mathbf{q})$ is linear with respect to probability distributions over the sets of pure strategies of players, it is a uniformly continuous function. The expected payoff of A when he chooses a pure strategy a_i and B chooses a mixed strategy \mathbf{q} is $M(a_i, \mathbf{q}) = \sum_{j=1}^n M(a_i, b_j) q_j$, and his expected payoff when he chooses a mixed strategy \mathbf{p} and B chooses a pure strategy b_j is $M(\mathbf{p}, b_j) = \sum_{i=1}^m p_i M(a_i, b_j)$. The set of all mixed strategies of A is denoted by P , and that of B is denoted by Q . P is an $m - 1$ -dimensional simplex, and Q is an $n - 1$ -dimensional simplex.

We call $v_A(\mathbf{p}) = \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q})$ the *guaranteed payoff* of A at \mathbf{p} . And we define v_A^* as follows,

$$v_A^* = \sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q})$$

²See Theorem 2.2.13 of [4].

This is a constructive version of the maximin payoff. Similarly, we call $v_B(\mathbf{q}) = \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q})$ the guaranteed payoff of player B at \mathbf{q} , and define v_B^* as follows,

$$v_B^* = \inf_{\mathbf{q}} \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q}).$$

This is a constructive version of the minimax payoff. For a fixed \mathbf{p} we have $\inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) \leq M(\mathbf{p}, \mathbf{q})$ for all \mathbf{q} , and so

$$\sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) \leq \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q}) \text{ for all } \mathbf{q}$$

holds. Then, we obtain $\sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\mathbf{p}, \mathbf{q}) \leq \inf_{\mathbf{q}} \sup_{\mathbf{p}} M(\mathbf{p}, \mathbf{q})$. This is rewritten as

$$v_A^* \leq v_B^*. \quad (1)$$

Define a function $\Gamma = (\mathbf{p}'(\mathbf{p}, \mathbf{q}), \mathbf{q}'(\mathbf{p}, \mathbf{q}))$ as follows;

$$p'_i(\mathbf{p}, \mathbf{q}) = \frac{p_i + \max(M(a_i, \mathbf{q}) - M(\mathbf{p}, \mathbf{q}), 0)}{1 + \sum_{k=1}^m \max(M(a_k, \mathbf{q}) - M(\mathbf{p}, \mathbf{q}), 0)},$$

$$q'_j(\mathbf{p}, \mathbf{q}) = \frac{q_j + \max(M(\mathbf{p}, \mathbf{q}) - M(\mathbf{p}, b_j), 0)}{1 + \sum_{k=1}^n \max(M(\mathbf{p}, \mathbf{q}) - M(\mathbf{p}, b_k), 0)}.$$

We assume the following condition;

Assumption 1. All sequences $((\mathbf{p}_n, \mathbf{q}_n))_{n \geq 1}$, $((\mathbf{p}'_n, \mathbf{q}'_n))_{n \geq 1}$ in $P \times Q$ such that $\max(M(a_i, \mathbf{q}_n) - M(\mathbf{p}_n, \mathbf{q}_n), 0) \rightarrow 0$, $\max(M(\mathbf{p}_n, \mathbf{q}_n) - M(\mathbf{p}_n, b_j), 0) \rightarrow 0$, $\max(M(a_i, \mathbf{q}'_n) - M(\mathbf{p}'_n, \mathbf{q}'_n), 0) \rightarrow 0$ and $\max(M(\mathbf{p}'_n, \mathbf{q}'_n) - M(\mathbf{p}'_n, b_j), 0) \rightarrow 0$ for all i and j are eventually close in the sense that $|(\mathbf{p}_n, \mathbf{q}_n) - (\mathbf{p}'_n, \mathbf{q}'_n)| \rightarrow 0$.

Under this assumption, we find

All sequences $((\mathbf{p}_n, \mathbf{q}_n))_{n \geq 1}$, $((\mathbf{p}'_n, \mathbf{q}'_n))_{n \geq 1}$ in $P \times Q$ such that $|\Gamma((\mathbf{p}_n, \mathbf{q}_n)) - (\mathbf{p}_n, \mathbf{q}_n)| \rightarrow 0$ and $|\Gamma((\mathbf{p}'_n, \mathbf{q}'_n)) - (\mathbf{p}'_n, \mathbf{q}'_n)| \rightarrow 0$ are eventually close in the sense that $|(\mathbf{p}_n, \mathbf{q}_n) - (\mathbf{p}'_n, \mathbf{q}'_n)| \rightarrow 0$.

Thus, Γ has sequentially at most one fixed point.

Summing up p'_i from 1 to m , for each i

$$\sum_{i=1}^m p'_i(\mathbf{p}, \mathbf{q}) = \frac{\sum_{i=1}^m p_i + \sum_{i=1}^m \max(M(a_i, \mathbf{q}) - M(\mathbf{p}, \mathbf{q}), 0)}{1 + \sum_{k=1}^m \max(M(a_k, \mathbf{q}) - M(\mathbf{p}, \mathbf{q}), 0)} = 1.$$

Similarly, summing up q'_j from 1 to n , for each j

$$\sum_{j=1}^n q'_j(\mathbf{p}, \mathbf{q}) = \frac{\sum_{j=1}^n q_j + \sum_{j=1}^n \max(M(\mathbf{p}, \mathbf{q}) - M(\mathbf{p}, b_j), 0)}{1 + \sum_{k=1}^n \max(M(\mathbf{p}, \mathbf{q}) - M(\mathbf{p}, b_k), 0)} = 1.$$

Let $\mathbf{p}'(\mathbf{p}, \mathbf{q}) = (p'_1, p'_2, \dots, p'_m)$, $\mathbf{q}'(\mathbf{p}, \mathbf{q}) = (q'_1, q'_2, \dots, q'_n)$. Then, $\Gamma = (\mathbf{p}'(\mathbf{p}, \mathbf{q}), \mathbf{q}'(\mathbf{p}, \mathbf{q}))$ is a uniformly continuous function from $P \times Q$ into itself. There are $m + n - 2$ independent vectors in $P \times Q$, and so $P \times Q$ is an $m + n - 2$ -dimensional space. Since it is a product of two simplices, it is a compact subset of a metric space. Therefore, Γ has a fixed point. Let $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ be the fixed point, and $\lambda = \sum_{k=1}^m \max(M(a_k, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}), 0)$, $\lambda' = \sum_{k=1}^n \max(M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, b_k), 0)$. Then,

$$\frac{\tilde{p}_i + \max(M(a_i, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}), 0)}{1 + \lambda} = \tilde{p}_i,$$

$$\frac{\tilde{q}_j + \max(M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, b_j), 0)}{1 + \lambda'} = \tilde{q}_j.$$

Thus, we have

$$\max(M(a_i, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}), 0) = \lambda \tilde{p}_i,$$

and

$$\max(M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, b_j), 0) = \lambda' \tilde{q}_j.$$

Since $M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) = \sum_{i=1}^m M(a_i, \tilde{\mathbf{q}})$, it is impossible that $\max(M(a_i, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}), 0) = M(a_i, \tilde{\mathbf{q}}) - M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) > 0$ for all i such that $\tilde{p}_i > 0$. Therefore, $\lambda = 0$, and we have $\sup_{\mathbf{p}} M(\mathbf{p}, \tilde{\mathbf{q}}) = M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$. Similarly, we obtain $\lambda' = 0$ and $\inf_{\mathbf{q}} M(\tilde{\mathbf{p}}, \mathbf{q}) = M(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$. Then,

$$v_B^* = \inf_{\mathbf{q}} \sup_{\mathbf{p}} M(\mathbf{p}, \tilde{\mathbf{q}}) \leq \sup_{\mathbf{p}} \inf_{\mathbf{q}} M(\tilde{\mathbf{p}}, \mathbf{q}) = v_A^*.$$

With (1) we obtain

$$v_A^* = v_B^*.$$

Therefore, the value of the game is determined at the fixed point of Γ .

		Player 2	
		X	Y
Player 1	X	1, -1	-1, 1
	Y	-1, 1	1, -1

TABLE 1. Example of game

Consider an example. See a game in Table 1. It is an example of the so-called Matching-Pennies game. Pure strategies of Player 1 and 2 are X and Y . The left side number in each cell represents the payoff of Player 1 and the right side number represents the payoff of Player 2. Let p_X and $1 - p_X$ denote the probabilities that Player 1 chooses, respectively, X and Y , and q_X and $1 - q_X$ denote the probabilities for Player 2. Denote the expected payoff of Player 1 by $M(p_X, q_X)$. Since we consider a zero-sum game, the expected payoff of Player 2 is $-M(p_X, q_X)$. Then,

$$M(p_X, q_X) = p_X q_X - (1 - p_X) q_X - p_X (1 - q_X) + (1 - p_X) (1 - q_X) = (2p_X - 1)(2q_X - 1)$$

Denote the payoff of Player 1 when he chooses X by $M(X, q_X)$, and that when he chooses Y by $M(Y, q_X)$. Similarly for Player B. Then,

$$M(X, q_X) = 2q_X - 1, \quad M(Y, q_X) = 1 - 2q_X, \quad -M(p_X, X) = 1 - 2p_X, \quad -M(p_X, Y) = 2p_X - 1.$$

And we have

When $q_X > \frac{1}{2}$, $M(X, q_X) > M(Y, q_X)$ and $M(X, q_X) > M(p_X, q_X)$ for $p_X < 1$,

When $q_X < \frac{1}{2}$, $M(Y, q_X) > M(X, q_X)$ and $M(Y, q_X) > M(p_X, q_X)$ for $p_X > 0$,

When $p_X > \frac{1}{2}$, $-M(p_X, Y) > -M(p_X, X)$ and $-M(p_X, Y) > -M(p_X, q_X)$ for $q_X > 0$,

When $p_X < \frac{1}{2}$, $-M(p_X, X) > -M(p_X, Y)$ and $-M(p_X, X) > -M(p_X, q_X)$ for $q_X < 1$.

Consider sequences $(p_X(m))_{m \geq 1}$ and $(q_X(m))_{m \geq 1}$, and let $0 < \varepsilon < \frac{1}{2}$, $0 < \delta < \varepsilon$. There are the following cases.

- (1) (a) If $p_X(m) > \frac{1}{2} + \delta$ and $q_X(m) > \frac{1}{2} + \delta$, or
- (b) $p_X(m) > \frac{1}{2} + \delta$ and $q_X(m) < \frac{1}{2} - \delta$, or

- (c) $p_X(m) < \frac{1}{2} - \delta$ and $q_X(m) < \frac{1}{2} - \delta$, or
 - (d) $p_X(m) < \frac{1}{2} - \delta$ and $q_X(m) > \frac{1}{2} + \delta$, or
 - (e) $p_X(m) > \frac{1}{2} + \delta$ and $\frac{1}{2} - \varepsilon < q_X(m) < \frac{1}{2} + \varepsilon$, or
 - (f) $p_X(m) < \frac{1}{2} - \delta$ and $\frac{1}{2} - \varepsilon < q_X(m) < \frac{1}{2} + \varepsilon$, or
 - (g) $\frac{1}{2} - \varepsilon < p_X(m) < \frac{1}{2} + \varepsilon$, and $q_X(m) > \frac{1}{2} + \delta$ or
 - (h) $\frac{1}{2} - \varepsilon < p_X(m) < \frac{1}{2} + \varepsilon$, and $q_X(m) < \frac{1}{2} - \delta$ or
- then, there exists no pair of $(p_X(m), q_X(m))$ such that $M(X, q_X(m)) - M(p_X(m), q_X(m)) \rightarrow 0$ and $-[M(p_X(m), Y) - M(p_X(m), q_X(m))] \rightarrow 0$.
- (2) If $\frac{1}{2} - \varepsilon < p_X(m) < \frac{1}{2} + \varepsilon$ and $\frac{1}{2} - \varepsilon < q_X(m) < \frac{1}{2} + \varepsilon$ with $0 < \varepsilon < \frac{1}{2}$, $M(X, q_X(m)) - M(p_X(m), q_X(m)) \rightarrow 0$, $M(Y, q_X(m)) - M(p_X(m), q_X(m)) \rightarrow 0$, $-[M(p_X(m), X) - M(p_X(m), q_X(m))] \rightarrow 0$ and $-[M(p_X(m), Y) - M(p_X(m), q_X(m))] \rightarrow 0$, then $(p_X(m), q_X(m)) \rightarrow (\frac{1}{2}, \frac{1}{2})$.

Therefore, the payoff functions satisfy Assumption 1.

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